14.1 Multivariable Functions

Up to now, we've been investigating functions that have ***only one*** input.

Examples: TR(q), MC(q), D(t), f(x), etc…

A ***multivariable*** function has more than one input.

Examples:

$C(h, p, x, y, z) = \text{course percentage}$

$A(P, r, t) = Pe^{rt}$

$BMI(w, h) = \frac{703w}{h^2}$

$TC(x, y) = 3x + 5y + 10$

$TR(x, y) = 8x + 6y$

Goal: To find and interpret derivatives of multivariable functions. And use them to find critical points.

Idea: Look at one variable at a time.

Ex) $z = \frac{x^2}{y^3}$

If $x = 2$, then

$z = \frac{4}{y^3} = 4y^{-3}$

$\frac{dz}{dy} = -12y^{-4}$

If $y = 2$, then

$z = \frac{x^2}{8} = \frac{1}{8}x^2$

$\frac{dz}{dx} = \frac{1}{4}x$

Aside: (You don't need to know this for this course, but I think it might help you visualize what is going on).

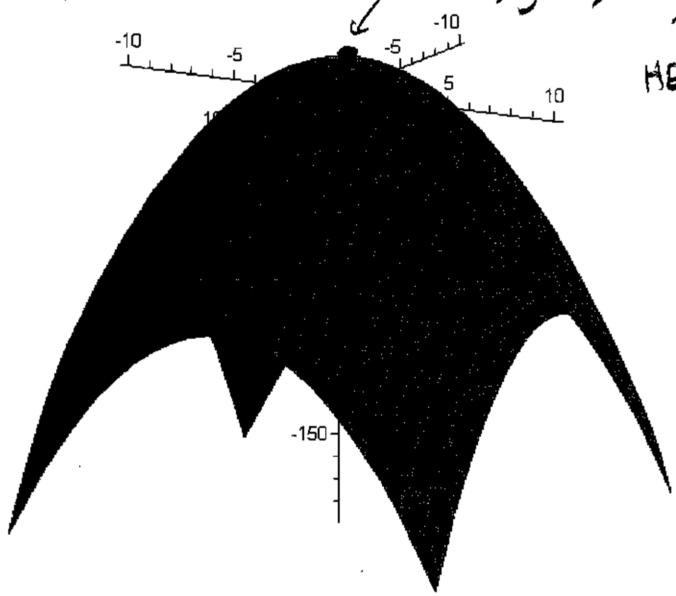
The graph of a 2-variable function is a *surface*, where the output is the height of the surface.

Example: (A "paraboloid")

$z = f(x, y) = 15 - x^2 - y^2$

x=0, y=0, z=15

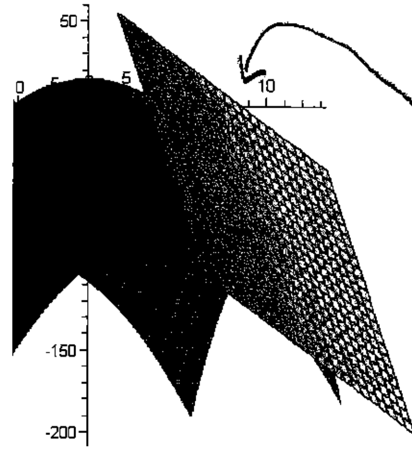
HEIGHT



Example: (A "plane")

$$z = g(x, y) = \underbrace{-14x - 8y}_{} + 80$$

A plane

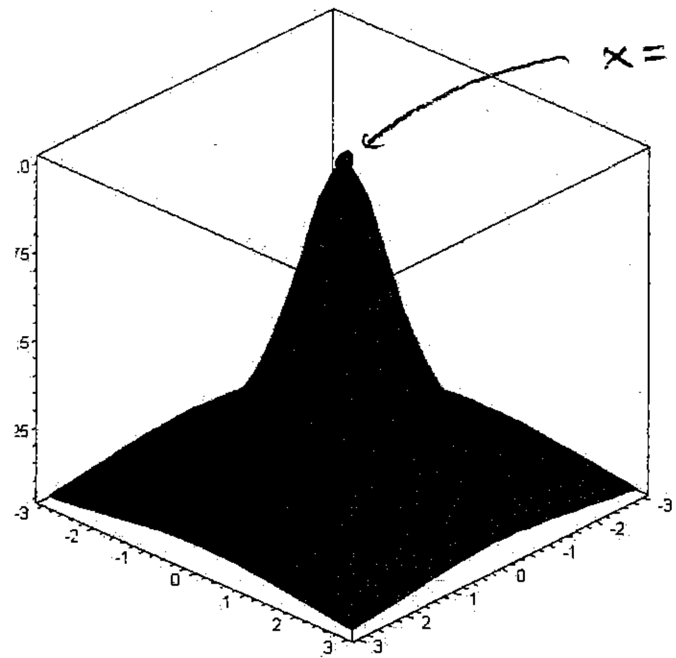


Example: $z = h(x, y) = \frac{1}{1+x^2+y^2}$

x=0, y=0

$z = \frac{1}{1+0^2+0^2} = 1$

height



Example:

$$f(x,y) = 4xy + y^2 - 3x - 5y$$

Find and simplify

$$\frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{[4(x+h)y + y^2 - 3(x+h) - 5y] - [4xy + y^2 - 3x - 5y]}{h}$$

$$\frac{\cancel{4xy} + 4hy + \cancel{y^2} - \cancel{3x} - 3h - \cancel{5y} - \cancel{4xy} - \cancel{y^2} + \cancel{3x} + \cancel{5y}}{h}$$

$$= \frac{4hy - 3h}{h} = 4y - 3$$

Let $h \to 0$

$$\boxed{\frac{\partial z}{\partial x} = f_x(x,y) = 4y - 3}$$

Find and simplify

$$\frac{f(x,y+h) - f(x,y)}{h}$$

$$\frac{[4x(y+h) + (y+h)^2 - \cancel{3x} - 5(y+h)] - [4xy + y^2 - \cancel{3x} - 5y]}{h}$$

$$\frac{\cancel{4xy} + 4xh + \cancel{y^2} + 2hy + h^2 - \cancel{5y} - 5h - \cancel{4xy} - \cancel{y^2} + \cancel{5y}}{h}$$

$$\frac{4xh + 2hy + h^2 - 5h}{h}$$

$$4x + 2y + h - 5$$

Let $h \to 0$

$$\boxed{\frac{\partial z}{\partial y} = f_y(x,y) = 4x + 2y - 5}$$

Short-cut:

$$f(x, y) = 4xy + y^2 - 3x - 5y$$

1. Use the derivative rules to find the derivative with respect to *x* (treat y like a constant).
2. Now use the rules to find the derivative will respect to y (treat x like a constant)

coeff. constant constant

$$z = 4xy + y^2 - 3x - 5y$$

coef coef coef

$$\frac{\partial z}{\partial x} = 4 \cdot 1 \cdot y + 0 - 3 \cdot 1 - 0$$

$$= 4y - 3$$

coeff. constant

$$z = 4xy + y^2 - 3x - 5y$$

coef. coef coef.

$$\frac{dz}{dy} = 4x \cdot 1 + 2y - 0 - 5 \cdot 1$$

$$= 4x + 2y - 5$$

Pretend you are skiing on the surface
$z = f(x, y) = 15 - x^2 - y^2$

1. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

2. Find their values when
   x = 7 and y = 4

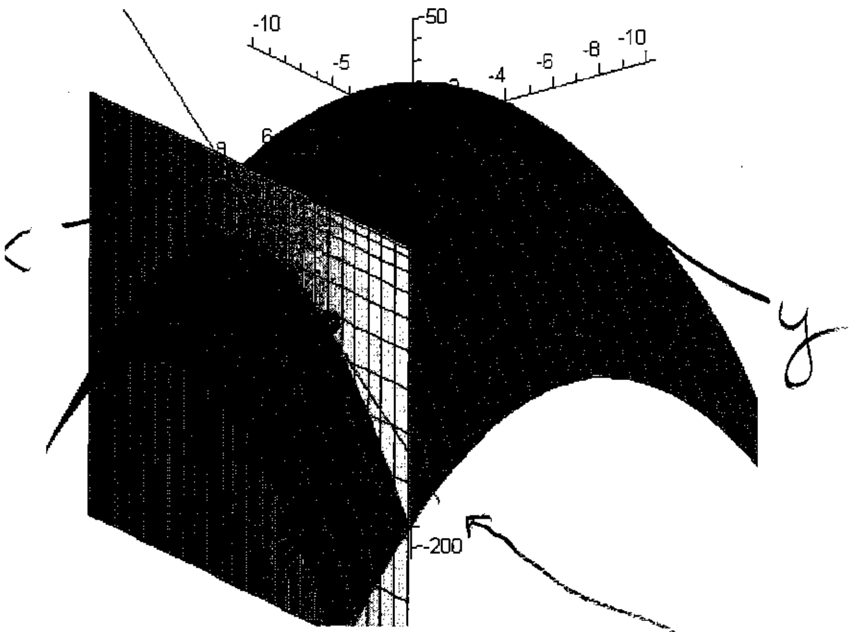
$$\frac{\partial z}{\partial x} = -2x$$

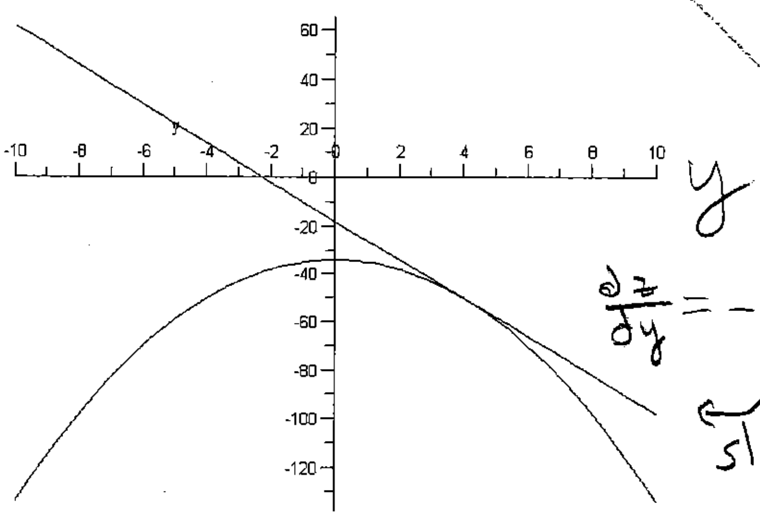
$$\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial z}{\partial x}(7,4) = -2(7) = -14$$

$$\frac{\partial z}{\partial y}(7,4) = -2(4) = -8$$

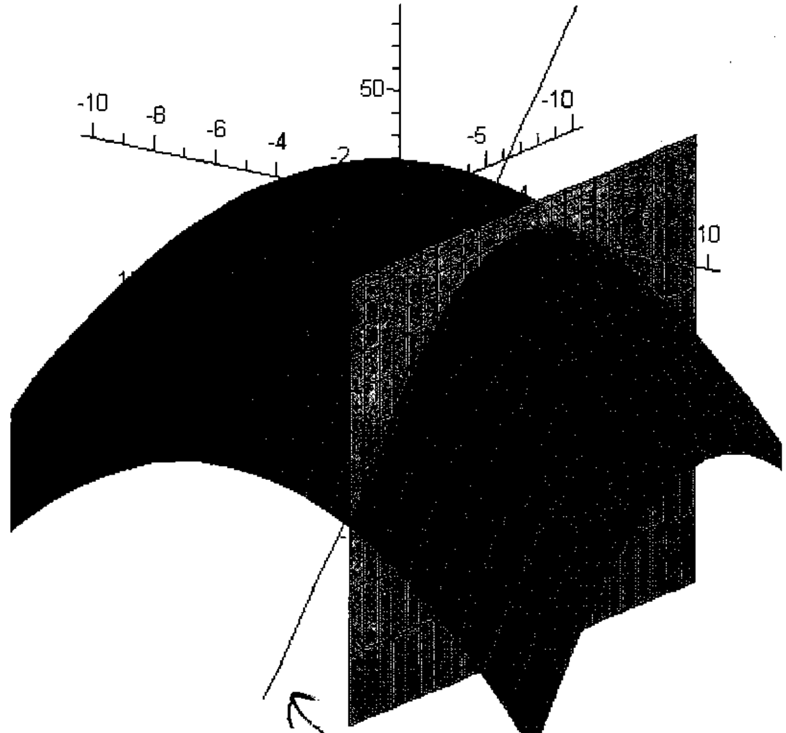
Aside: Graphical Interpretations

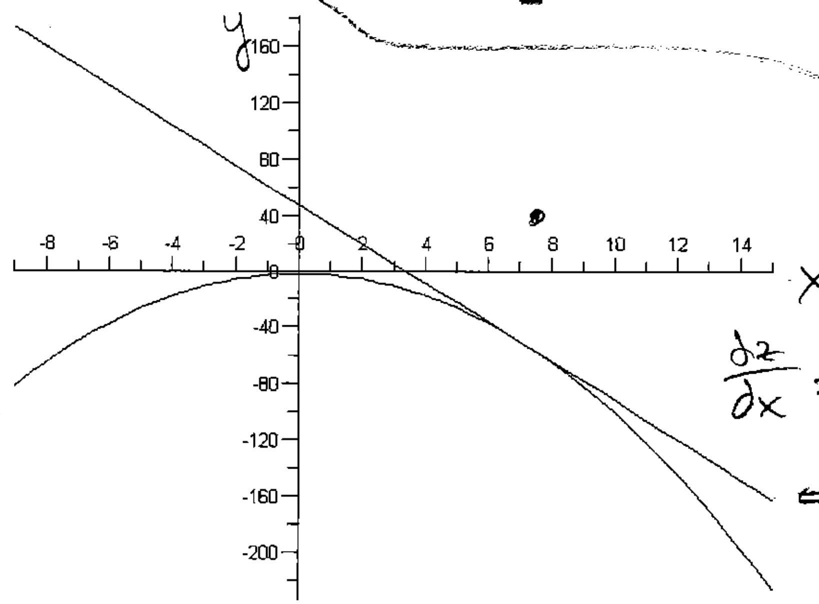




$\frac{\partial z}{\partial y} = -8$

slope





$\frac{\partial z}{\partial x} = -14$

slope

Recall: Before we found the derivative short-cuts, we discussed how:

1. Given a function $y = f(x)$.
2. Simplify the general formula for the slope of the secant from x to $x + h$
$$\frac{f(x+h) - f(x)}{h}$$
3. Let $h \to 0$, to get
$$\frac{dy}{dx} = f'(x) = \text{slope of tangent}$$

Partial Derivatives

For multivariable functions, we are going to fix all the input variables except one (treat them like constants). Then we'll compute the derivative with respect to that one variable function.

Given $z = f(x, y)$

With respect to x as variable: Fix y!
$$\frac{f(x+h, y) - f(x, y)}{h}$$
Let $h \to 0$, to get
$$\frac{\partial z}{\partial x} = f_x(x, y)$$

With respect to y as variable: Fix x!
$$\frac{f(x, y+h) - f(x, y)}{h}$$
Let $h \to 0$, to get
$$\frac{\partial z}{\partial y} = f_y(x, y)$$